Affine plane curves

Recall that an irreducible affine plane curve $C \subseteq \mathbb{A}^2$ corresponds to an ideal $(f) \subseteq k[x,y]$ where f is irreducible.

However, we want to consider reducible curves w/ multiple components, so we define them more generally.

Let $f, g \in k[x,y]$. Note that $(f) = (g) \iff f = \lambda g$, $0 \neq \lambda \in k$.

Def: An affine plane curve is an equivalence class of nonconstant polynomials in $k[x,y]$ under the equivalence relation $f \sim g \iff f = \lambda g$.

In this case, f and g are equivalent. The degree of the curve is the degree of the defining polynomial.

Note: We are now allowing reducible polynomials, even those that generate nonradical ideals. i.e. even though $V(x) = V(x^2)$, x and x^2 are different curves.

A curve of degree one is a line.

If $f = \prod f_i^{e_i}$, f_i irreducible, then the f_i are the components of f and e_i the multiplicity of f_i.

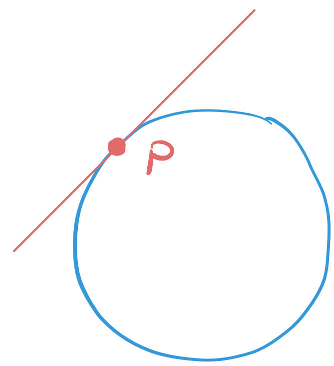
If $e_i = 1$, f_i is a simple component.

Tangent lines via calculus

How do we find the tangent line to a plane curve at a point?

Calculus method:

Let $f = y^2 + x^2 - 1 = 0$, $P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$



$f_x := \frac{df}{dx} = 2x$, $f_y = 2y$

$f_x(P) = -\sqrt{2}$, $f_y(P) = \sqrt{2}$

Tangent line: $-\sqrt{2}\left(x + \frac{\sqrt{2}}{2}\right) + \sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right) = 0 \Rightarrow y = x + \sqrt{2}$.

Def: P is a simple point or smooth point if $f_x(P) \neq 0$ or $f_y(P) \neq 0$.

In this case, we know from calculus that

$$f_x(P)(x-a) + f_y(P)(y-b) = 0$$

is the tangent line to f at $P = (a, b)$.

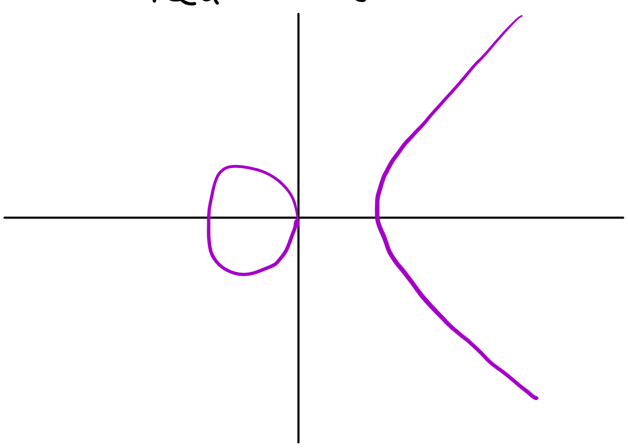
A point that's not simple is called multiple or singular.

If all the points on f are simple, f is nonsingular or smooth.

Ex: (over $\mathbb{C}$)

1.) $f = y^2 - x^3 + x$

Real locus:



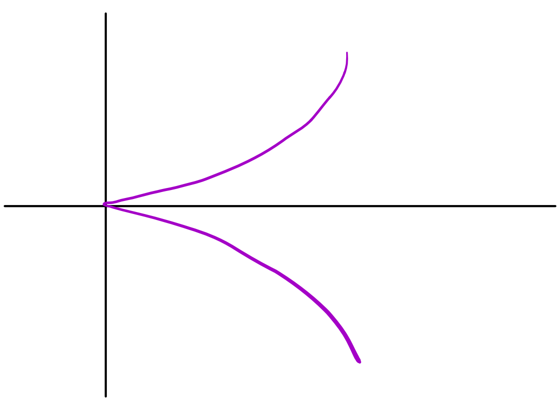
$f_x = 1 - 3x^2 = 0 \Rightarrow x = \frac{\pm\sqrt{3}}{3}$

$f_y = 2y = 0 \Rightarrow y = 0$

$\left(\frac{\pm\sqrt{3}}{3}, 0\right) \notin V(f)$, so f is nonsingular

Note: lowest degree term is x, which is the tangent line at $(0,0)$.

2.) $g = y^2 - x^3$



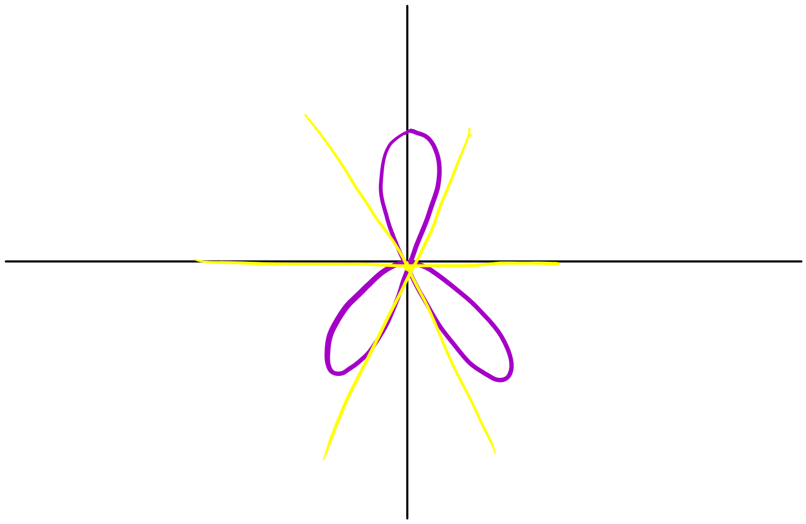
$g_x = -3x^2$, $g_y = 2y$, so g is singular at $(0,0)$

Note: lowest degree term $= y^2$, and y is "tangent line" at $(0,0)$.

3.) $h = (x^2+y^2)^2 + 3x^2y - y^3$

$h_x = 2x(2x^2 + 2y^2 + 3y)$

$h_y = 4y(x^2+y^2) + 3(x^2 - y^2)$



Can check: these are both $0 \iff x = y = 0$

Note: lowest degree term

$$= 3x^2y - y^3$$
$$= y(3x^2 - y^2)$$
$$= y(\sqrt{3}x - y)(\sqrt{3}x + y)$$

look like 3 "tangent lines"

Need a more algebraic approach in order to understand what's going on here:

Homogeneous polynomials

Def: $F \in k[x_1, \dots, x_n]$ is homogeneous or a form of deg d if it can be written as the sum of monomials of degree d.

We can dehomogenize F (w/ respect to x_n) by setting

$$f = F(x_1, \dots, x_{n-1}, 1) \in k[x_1, \dots, x_{n-1}].$$

This is the same as taking the image of F in $\frac{k[x_1, \dots, x_n]}{(x_n - 1)} \cong k[x_1, \dots, x_{n-1}]$

If $f \in k[x_1, \dots, x_n]$ is any polynomial of deg d, we can write

$$f = f_0 + f_1 + \dots + f_d,$$

where f_i is a (possibly 0) form of degree i, $f_d \neq 0$.

Def: The homogenization of f is defined

$$F = x_{n+1}^{d} f_0 + x_{n+1}^{d-1} f_1 + \dots + f_d \in k[x_1, \dots, x_{n+1}]$$

Then F is a form of degree d.

Ex: If $F = x^2 z + y^2 z + x z^2 + z^3$ (a form of degree 3).

Then we dehomogenize with respect to z and get

$x^2 + y^2 + x + 1$, a polynomial of degree 2. So if we homogenize again, we get $x^2 + y^2 + xz + z^2 \neq F$.

Note: Homogenization commutes w/ multiplication but not addition. (check)

Prop: If $F \in k[x,y]$ is homogeneous, k algebraically closed, then F factors into a product of linear forms.

Pf: Write $F = y^r G$, $r \geq 0$, s.t. y doesn't divide G.

Then the dehomogenization of G is $\alpha \prod_i (x - \lambda_i)$, $\alpha, \lambda_i \in k$.

$$\Rightarrow G = \alpha \prod_i (x - \lambda_i y) \Rightarrow F = \alpha y^r \prod_i (x - \lambda_i y). \square$$

Multiplicities

Let f be a plane curve, $P=(0,0)$.

Write $f = f_m + f_{m+1} + \dots + f_{m+n}$, where f_i is a form of deg i,

and $f_m \neq 0$.

Def: The <u>initial form</u> of f at $P=(0,0)$ is $\mathrm{in}(f) := f_m$.

The <u>multiplicity</u> of f at $P=(0,0)$ is $m_P(f) := \deg f_m = m$.

Remark: $(0,0) \in V(f) \iff m_{(0,0)}(f) > 0$.

Prop: $P=(0,0)$ is a simple point of $f \iff m_P(f) = 1$.

Pf: For $i \geq 0$, $(f_i)_x$ and $(f_i)_y$ are 0 or forms of degree $i-1$. So $f_x(P) = (f_1)_x(P) + 0$ and $f_y(P) = (f_1)_y(P) + 0$.

If $f_1 = ax + by$, then $f_x(P) = f_y(P) = 0 \iff a = b = 0$. $\square$

Easy to check: In this case, f_1 is the tangent line to f at P.

Set $m = m_P(f)$. We can write

$$f_m = \prod_i L_i^{r_i}, \text{ where the } L_i \text{ are distinct lines.}$$

Def: The L_i are tangent lines to f at $P=(0,0)$.

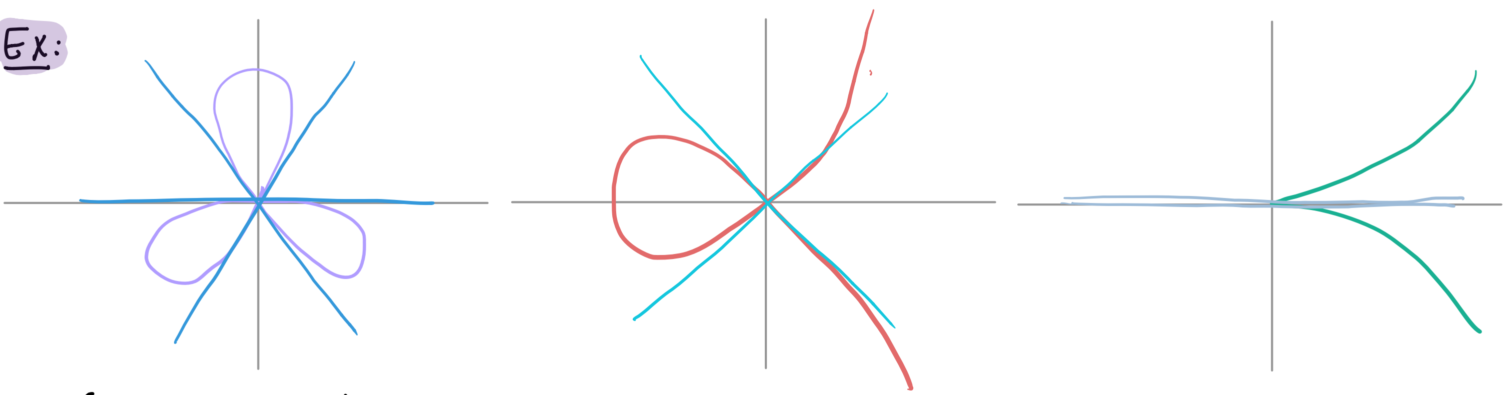
f_m is called the tangent cone to f at the origin.

r_i is the multiplicity of the tangent line.

If f has $m = m_P(f)$ distinct tangents at P (i.e. $r_i = 1$), or, equivalently, the tangent cone is reduced, then P is an ordinary point of f.

An ordinary double point (i.e. point of mult. 2) is a node.

Ex:



$\mathrm{in}(f) = y(\sqrt{3}x+y)(\sqrt{3}x-y)$
$(0,0)$ is an ordinary triple point

$\mathrm{in}(f) = (x+y)(x-y)$
$\Rightarrow (0,0)$ is a node

$\mathrm{in}(f) = y^2$
$(0,0)$ is a singular point, but not ordinary

Note: You can check that these are in fact tangent lines, using the standard calculus definition, by parametrizing the curve and finding the tangent line along each branch.

Remark: $\mathrm{in}(fg) = \mathrm{in}(f)\,\mathrm{in}(g)$.

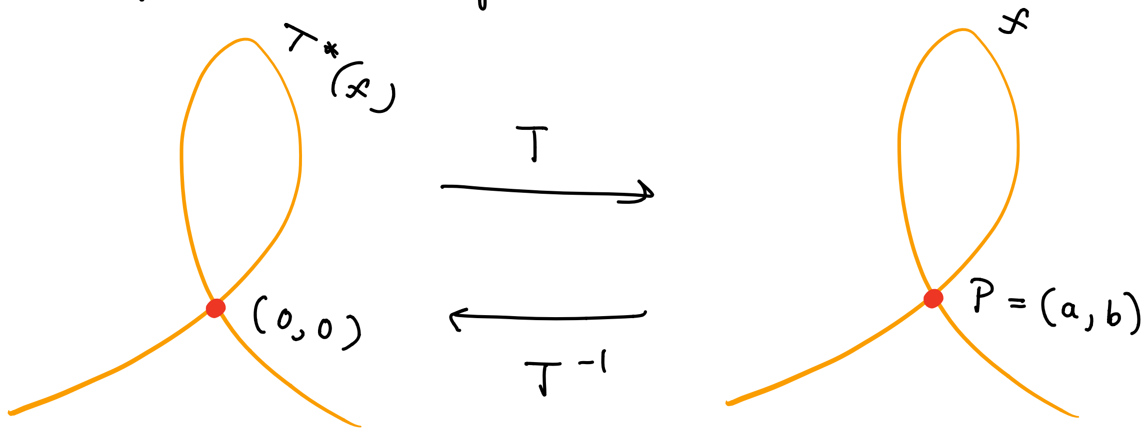
Thus, if $f = \prod h_i^{e_i}$ is the factorization into irreducible components, $m_P(f) = \sum e_i m_P(h_i)$.

Points on curves away from the origin

Let $P = (a,b)$, and $T: \mathbb{A}^2 \to \mathbb{A}^2$ defined

$$T(x,y) = (x+a,\ y+b).$$

Then $T^*(f) = f(x+a,\ y+b)$.



Define $m_P(f) = m_{(0,0)}\left(T^*(f)\right)$.

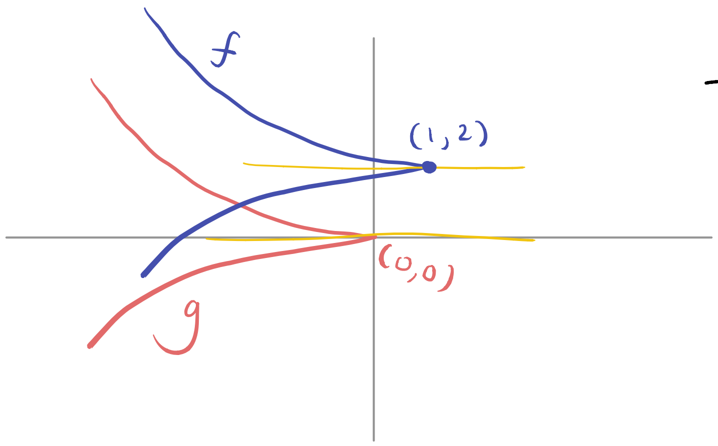
If $L_i = \alpha x + \beta y$ is a tangent line to $T^*(f)$ at $(0,0)$ w/ multiplicity e_i, then $\alpha(x-a) + \beta(y-b)$ is a tangent line to f at P.

In other words, we can do an affine change of coordinates so that the point we're interested in is at the origin.

Ex: Let $f = x^3 + y^2 - 3x^2 - 4y + 3x + 3$

Then $f_x = 3x^2 - 6x + 3$ $\Rightarrow$ f is singular if $x = 1$, $y = 2$.
$f_y = 2y - 4$ and $(1,2) \in V(f)$.

So we can look at $g = f(x+1, y+2) = y^2 + x^3$, which is now singular at the origin.



Thus, $m_{(1,2)}(f) = m_{(0,0)}(g) = 2$ and the tangent line is $y - 2$ and has multiplicity 2.

Tangent spaces + local rings

We'll soon see the relationship between the local geometry of a curve and the local ring at a point, but here's a preview:

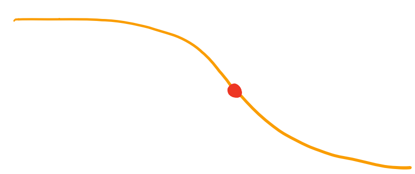
Let $f \in k[x,y]$ be a curve and P a point of f. Let $\mathcal{O}_P(f)$ be the local ring at P and m_P the max'l ideal.

Def: The cotangent space of f at P is the $\mathcal{O}_P/m_P \cong k$ -vector space m_P/m_P^2.

Ex: Let $f = x - y^3$, $P = (0,0)$.

Then $\mathcal{O}_P = \left\{ \frac{g}{h} \;\middle|\; h \text{ has a nonzero constant term},\ f, g \in \Gamma(V(f)) \right\}$

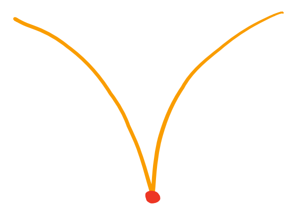
$m_P = (\bar{x}, \bar{y})$, $m_P^2 = (\bar{x}^2, \bar{y}^2, \overline{xy})$



But $\bar{x} = -\bar{y}^3$, so $m_P = (\bar{y}^3, \bar{y}) = (\bar{y})$.

Thus, m_P/m_P^2 has k-basis $\bar{y}$, so it's one-dimensional.

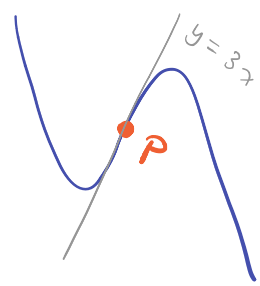
Ex: Let $f = x^2 - y^3$. Then m_P has k-basis $\bar{x}, \bar{y}, \bar{y}^2, \dots$, but $\bar{y}^2, \bar{y}^3, \dots \in m_P^2$, so $\dim_k m_P/m_P^2 = 2$.



Def: The <u>Zariski tangent space</u> of f at P is the dual of m_P/m_P^2 (as a k-vector space). i.e. it's the space of k-linear maps $m_P/m_P^2 \longrightarrow k$.

How does this give us a tangent space geometrically?

Ex: $f = y - 3x + x^3$, $P = (0,0)$



$m_P = (x, y)$, but $y = 3x - x^3$,

so it has basis $x, x^2, x^3, \dots$

so m_P/m_P^2 has dimension k. Moreover, $\bar{y} - \overline{3x} + \bar{x}^3 = 0$,

but $\bar{x}^3 = 0$, so $\bar{y} = \overline{3x}$.

The linear maps $\mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ send $x \mapsto a$, $y \mapsto 3a$

for each $a \in k$. This describes an embedding $\begin{array}{l} \mathbb{A}^1 \to \mathbb{A}^2 \\ a \mapsto (a, 3a) \end{array}$

(If you want to know more details, this would be a good final project topic!)